Optimum Skewed Redundant Inertial Navigators

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The problem of proper placement of the inertial sensors to optimize system performance is important to system designers. This is especially true of redundant strapdown systems that employ more sensors than the conventional, mutually orthogonal sets of three. In systems designed for a free fall environment and with no preferred direction, such as for spacecraft attitude reference, the sensor input axes should divide the three-space equally, and can thus be viewed as being normal to the faces of regular polyhedra.[1,2] In Earth-bound inertial navigators, the effect of gravity-dependent sensor errors and the generally reduced effect of azimuth errors on navigation accuracy significantly alter the situation. Both effects tend to decrease sensor elevation angles in an optimized system. Formulas are derived and curves are drawn for optimum sensor elevation and azimuth angles vs a g-sensitive sensor error parameter, and a mission relative azimuth error parameter. Tetrad, pentad, and hexad arrays are analyzed, affording a dramatic improvement in accuracy as well as autonomous fault isolation and/or detection capability.

Redundancy and Fault Isolation or Detection

THE number and proper orientation of the inertial sensors (gyros and accelerometers) in an inertial navigator designed for use in a gravity environment depend on the mission requirements. Two of the most important requirements are reliability and accuracy. The desired reliability level dictates the number of sensors which must be used in a system.

It is well-known that three noncoplanar sensors are necessary to provide full three-axis information in our 3-space. Addition of a fourth sensor not aligned with any of the other three to complete a tetrad, provides fault detection capability but is insufficient to provide fault isolation; that is, a fault can be detected by noting a disagreement among the outputs of the sensors. The amount of disagreement which can be tolerated is specified by system accuracy requirements.

The addition of a fifth sensor, no three of which are coplanar, completes a pentad which can provide positive fault isolation capability by using a voting technique among the ten triads. That is, assuming failure of a single sensor, the four triads not involving the faulty sensor will continue to show agreement, while the other six which involve the faulty sensor will not. A detailed discussion of fault isolation and of parity equations used to implement the voting technique is not intended here. This subject has been discussed elsewhere.[1,2]

The addition of a sixth sensor can, of course, provide two levels of fault isolation redundancy, if desired. Although not required for fault isolation, the addition of a sixth sensor to form a hexad does, however, provide greater accuracy and more reliable single fault isolation capability since, in effect, the outputs from 20 triads are compared and averaged in the parity and processing equations.

To span the foreseeable redundancy requirements in Earth inertial navigators, optimum tetrad, pentad, and hexad configurations are proposed and analyzed.

Inertial System Accuracy

Expected system accuracy is, of course, statistical in nature and improves with the number of sensors employed. Most previous analyses,[1] analyzed system accuracy and optimum sensor placement under the assumption that sensed information (angular displacement or acceleration) is equally important from all directions and that sensor accuracy is acceleration or gravity independent. Under these assumpions it was shown that sensors whose input axes are placed normal to the faces of regular polyhedra which divide the 3-space into equal regions comprise optimum systems. In such systems variance is minimized. System variance or mean radial variance is determined by integrating and averaging the variance over the entire 3-space and is found to be

$$\bar{\sigma}^2 = (\sigma_p{}^2 + \sigma_q{}^2 + \sigma_r{}^2)/3 \tag{1}$$

where p, q, and r are convenient mutually orthogonal axes; that is, $\bar{\sigma}^2$ is one third the trace of the system covariance matrix.

It is shown further in Ref. 1 that for those conditions, p sensors, $p \geqq 3$, equally spaced around a cone with central half angle, $\theta = \cos^{-1} 1/(3)^{1/2}$ comprise an optimum system in that system variance is minimized. Other possibilities for minimum variance systems are those with sensors equally spaced on cones so that

$$\sum_{i=1}^{k} \frac{\cos^2 \theta_i}{k} = \frac{1}{3}$$

References 2 and 3 are concerned with the analysis and design of a dodecahedron hexad system.

Azimuth Effect on Accuracy

In inertial navigation systems which are designed to operate in a gravity environment and whose primary concern is that of horizontal position, say latitude and longitude, there are two additional important criteria which must be considered. The first criterion is that system azimuth, q, axis errors have a different, generally lesser, effect on horizontal position accuracy than errors in the level, p and r axes. In particular it is easily shown that an azimuth gyro bias drift error, σ_q, propagates into a cross-range error, σ_c approximately according to $\sigma_c = \sigma_q Vt^2$, whereas a roll axis gyro bias drift, σ_r, propagates according to $\sigma_c = \sigma_r Rt$, where V is speed, R is Earth radius, and t is time.

Hence the relative azimuth effect,

$$M^{-1} = \frac{\sigma_q}{\sigma_r} = \frac{Vt^2}{Rt} = \frac{B}{R}$$

depends approximately on the distance traveled, B, compared with Earth radius. For typical long-range aircraft missions of say, 1000–2000 miles, M would vary from $4 > M > 2$. To account for M, we define Navigation Variance,

$$\sigma_N{}^2 = (\sigma_p{}^2 + \sigma_q{}^2/M^2 + \sigma_r{}^2)/3 \tag{2}$$

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For sensors with variances, σ_i^2, we find[1] $\sigma_p^2 = \sigma_r^2 \sim \sigma_i^2/S^2\theta$ and $\sigma_q^2 \sim \sigma_i^2/2C^2\theta$. Thus the optimum, θ_0, which minimizes $\sigma_N^2 \sim \sigma_i^2(2/S^2\theta + 1/2M^2C^2\theta)$ is

$$\theta_0 = \tan^{-1}(2M^2)^{1/2} = \cos^{-1}(2M^2+1)^{-1/2} \quad (3)$$

Hence θ_0 increases with M.

Gravity-Dependent Effects

Gravity or acceleration dependence of sensor errors is the second important sensor placement criterion which affects optimum system design. The accuracy characteristics of conventional displacement gyros are generally measured in terms of a g-insensitive standard deviation drift rate, σ_0 (deg/hr) and a g-sensitive term σ_g (deg/hr/g) whose effect varies with the cosine of the elevation angle of the output axis (OA). Thus, a gyro with an input axis (IA) elevation angle $(\pi/2-\theta)$ can have a maximum OA elevation angle of θ. Effects such as anisoelastic and cross-coupling errors are comparatively small in a 1-g environment and are not considered here.

For uncorrelated g-sensitive and g-insensitive errors the resultant variance $\sigma^2(\theta)$ of a gyro whose IA has an elevation angle $(\pi/2-\theta)$ or consequently whose OA can have a maximum elevation angle θ, can be expressed as

$$\sigma^2(\theta) = \sigma_0^2 + \sigma_g^2 \cos^2\theta$$

Defining the ratio $\sigma_g/\sigma_0 = K$, we can write

$$\sigma^2(\theta) = \sigma_0^2(1+K^2C^2\theta) \quad (4)$$

where throughtout we define, $C\theta \doteq \cos\theta$, and $S\theta \doteq \sin\theta$. Values of K depend on gyro design and build variations. For representative MIG-type gyros such as the Honeywell GG1009 or the GG8001, values for K generally range from, $1 < K < 3$.

For this reason, in conventional orthogonal triad systems, the two level (p and r axis) gyros are usually oriented with their output axes vertical so that the g-sensitive mass unbalance effects along the IA and spin axes (SA) are virtually eliminated. The azimuth (q) gyro with its IA nominally vertical thus experiences the full g-sensitive effect so that $\sigma_q^2 = (1+K^2)\sigma_0^2$ and $\sigma_r^2 = \sigma_p^2 = \sigma_0^2$.

It has been noted previously that the effect on navigation variance, σ_N^2, of the azimuth gyro, σ_q^2, is reduced by a factor $1/M^2$. Thus when the two effects are combined, the resulting navigation variance is

$$\sigma_N^2 = (\sigma_p^2 + \sigma_q^2/M^2 + \sigma_r^2)/3 = \sigma_0^2\left(1 + \frac{1+K^2}{M^2} + 1\right)\Big/3$$

Hence if M^2 is approximately equal to $1+K^2$ the error effect of all three gyros is equalized and a sensor system results which is nearly ideal in that the gyros are oriented so that the navigation variance is minimized.

The extension of this effect, and of optimum sensor placement which minimizes navigation variance, to skewed (nonorthogonal), redundant ($N > 3$), sensor systems constitutes the principal result of this paper.

Optimization Technique

When there is more than one measure of a desired component, it can be shown that the linear combination which results in minimum component variance is that in which coefficients, a_i, of the various measures are inversely proportional to their variances; that is, given n measures, r_i of r, let a_i be coefficients of r_i so that $r = a_1r_1 + a_2r_2 + \cdots + a_nr_n$. Given $\sigma^2(r_i) = \sigma_i^2$, then

$$a_i = 1/\sigma_i^2/(1/\sigma_1^2 + 1/\sigma_2^2 + \cdots + 1/\sigma_n^2)$$

For the case $n = 2$ which covers all cases in these analyses

$$a_1 = 1/\sigma_1^2/(1/\sigma_1^2 + 1/\sigma_2^2) = \sigma_2^2/(\sigma_1^2 + \sigma_2^2) \quad (5a)$$

$$a_2 = 1/\sigma_2^2/(1/\sigma_1^2 + 1/\sigma_2^2) = \sigma_1^2/(\sigma_1^2 + \sigma_2^2) \quad (5b)$$

The resulting minimum variance of r is

$$\sigma_r^2 = 1/(1/\sigma_1^2 + 1/\sigma_2^2) \quad (6)$$

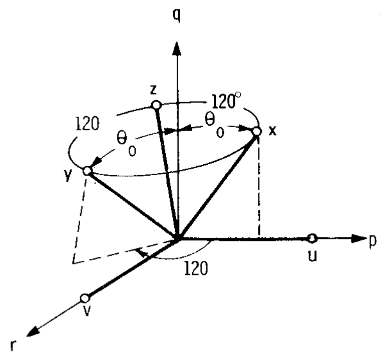

Fig. 1 Pentad input axes.

In the analyses, measures of p, q, and r are determined from triads or pairs of sensors to be optimally combined.

Pentad Analysis

The five-sensor pentad array is the minimum system which permits autonomous fault detection and isolation for fail-operational system capability, which is the principal reason for the use of redundant configurations. Fault isolation capability implies that no three sensors can be coplanar. If three are coplanar, only two measures of information remain along the axis normal to that plane. If one of those two sensors fails, there is no way of autonomously determining which of the two is yielding false information. Thus, a situation exists which, in effect, is the same as that for a tetrad. This implies that, for the pentad, two sensors may have their OA's nominally vertical, and IA's horizontal. It is presumed here that information is equally important in the two level axes which in inertial navigators is a good assumption. Thus symmetry in the pr plane is prescribed. This implies that for the pentad and hexad, sensors are equally spaced around the nominally vertical, q, axis. There thus remains only the task of determining the optimum, θ_0, which yields minimum navigation variance, σ_N^2. Referring to Fig. 1, let sensors u, v, x, y, and z be placed as shown

$$u = p_1, \; v = r_1$$

$$x = c\theta q + S\theta p_2$$

$$y = C\theta q - (S\theta p_2/2) + (3)^{1/2}S\theta r_2/2$$

$$z = C\theta q - (S\theta p_2/2) - (3)^{1/2}S\theta r_2/2$$

Combining p_1 and p_2, r_1, and r_2 optimally, we find

$$p = [u + S\theta(2x-y-z)/2(1+K^2C^2\theta)]/[1+3S^2\theta/2(1+K^2C^2\theta)] \quad (7a)$$

$$r = [v + (3)^{1/2}S\theta(y-z)/2(1+K^2C^2\theta)]/[1+3S^2\theta/2(1+K^2C^2\theta)] \quad (7b)$$

$$q = (x+y+z)/3C\theta \quad (7c)$$

and

$$\sigma_p^2 = \sigma_r^2 = \sigma_0^2/[1+3S^2\theta/2(1+K^2C^2\theta)]$$

$$\sigma_q^2 = (1+K^2C^2\theta)/3C^2\theta$$

We minimize σ_N^2 by setting $d\sigma_N^2/d\theta = 0$. Solving, we find

$$C^2\theta_0 = 5/[6M(1+K^2)^{1/2} + 3 - 2K^2] \quad (8a)$$

Figure 2 shows curves for θ_0.

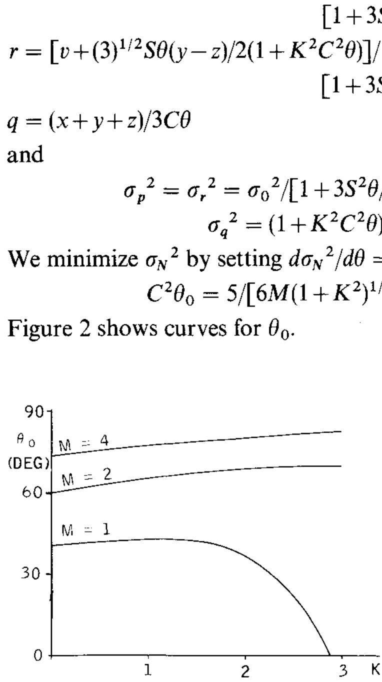

Fig. 2 θ_0 for pentad.

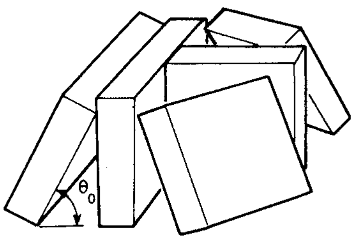

Fig. 3 Pentad configuration.

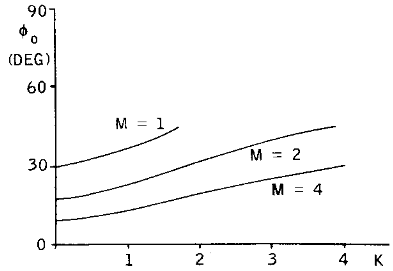

Fig. 5 ϕ_0 for tetrad.

Values of M and K for which $C^2\theta = \frac{1}{3}$ are particularly significant because the three elevated sensors (x, y, and z), form an orthogonal triad which when superimposed onto a horizontal orthogonal pair, u and v, comprise an attractive optimum pentad system pictured in Fig. 3. Values of M and K for which $C^2\theta = 1/3$, satisfy $M = (6+K^2)/3(1+K^2)^{1/2}$. For instance, $K = (3)^{1/2}$, $M = \frac{3}{2}$ are one such pair. Also for this pendad, $M^2 = 1+K^2$ for $M^2 = \frac{15}{4}$, $K^2 = \frac{11}{4}$. Other θ_0 values of interest can be analyzed similarly.

Tetrad Analysis

Optimized tetrads are considered for those applications where autonomous fault isolation is not a requirement but require the advantages of fault detection and improved accuracy afforded by the fourth sensor and by optimal sensor placement. The tetrad analysis is somewhat more complex than the pentad and hexad in that two interdependent placement variables or degrees of freedom, θ and ϕ, are necessary. Expressions for optimums $\phi_0(K, M)$ and $\theta_0(K, M)$ are derived which are to be used in system design for a given mission azimuth parameter M, and a g-sensitive sensor parameter, K. These are plotted in Figs. 4 and 5.

Let four sensors be placed with input axes u and v in the pr plane, x and y in the qr plane as shown on Fig. 6. Thus

$$u = C\phi p + S\phi r, \quad x = C\theta q + S\theta r$$
$$v = C\phi p - S\phi r, \quad y = C\theta q - S\theta r$$

solving for p, q, and r in each pair

$$p = (u+v)/2C\phi, \quad q = (x+y)/2C\theta; \quad \sigma_p{}^2 = \sigma_0{}^2/2C^2\phi$$
$$\sigma_q{}^2 = (1+KC^2\theta)\sigma_0{}^2/2C^2\theta$$
$$r_1 = (u-v)/2S\phi, \quad r_2 = (x-y)/2S\theta; \quad \sigma_{r_1}{}^2 = \sigma_0{}^2/2S^2\phi$$
$$\sigma_{r_2}{}^2 = (1+K^2C^2\theta)\sigma_0{}^2/2S^2\theta$$

Now, as discussed in the previous section, the coefficients a_1 and a_2 in $r = a_1r_1 + a_2r_2$, which minimize $\sigma_r{}^2$ are

$$a_1 = S_\phi{}^2/[S_\phi{}^2 + S_\theta{}^2/(1+K^2C^2\theta)]$$
$$a_2 = S_\theta{}^2/(1+K^2C^2\theta)/[S_\phi{}^2 + S_\theta{}^2/(1+K^2C^2\theta)]$$

Hence

$$r = [S\phi(u-v) + S\theta(x-y)]/\,(1+K^2C^2\theta)/2[S_\phi{}^2 + S_\theta{}^2/(1+K^2C^2\theta)]$$

Now navigation variance, $\sigma_N{}^2 = (\sigma_p{}^2 + \sigma_q{}^2/M^2 + \sigma_r{}^2)/3$. From the previous equation

$$f(C^2\theta, C^2\phi) = 6\sigma_N{}^2/\sigma_0{}^2 = 1/C_\phi{}^2 + (1+K^2C^2\theta)/M^2C^2\theta + (1+K^2C^2\theta)/[S^2\phi(1+K^2C^2\theta)+S^2\theta]$$

Let

$$C_\theta{}^2 = x, \quad S_\theta{}^2 = 1-x, \quad C_\phi{}^2 = y, \quad S_\phi{}^2 = 1-y$$

Thus

$$f(x, y) = (1/y)K^2/M^2 + 1/M^2x + 1/[1-y+(1-x)/(1+K^2x)]$$

To minimize $f(x, y)$ with respect to both θ and ϕ, set $\partial f/\partial x = 0$, $\partial f/\partial y = 0$ and solve simultaneously

$$\partial f/\partial y = -1/y^2 + 1/[1-y+(1-x)/(1+K^2x)]^2 = 0 \qquad \text{(9a)}$$
$$y = 1/2 + (1-x)/2(1+K^2x)$$

$$\frac{\partial f}{\partial x} = -1/M^2x^2 + (1+K^2)/[1-x+(1-y)(1+K^2x)]^2 = 0 \qquad \text{(9b)}$$
$$y = 1 + [1+M(K^2+1)^{1/2}]x/(1+K^2x)$$

setting Eq. (9a) = Eq. (9b) and solving for x

$$x = C^2\theta_0 = 1/[M(K^2+1)^{1/2} - (K^2-1)/2] \qquad \text{(8b)}$$

Equation (8b) for $C^2\theta_0$ is the same for the hexad and for all cases of an even number of sensors, half horizontal and half at an angle θ_0 with the vertical. The corresponding solution for y is obtained from Eq. (9a)

$$C^2\phi_0 = \tfrac{1}{2}[1 + (1-C^2\theta_0)/(1+K^2C^2\theta_0)] \qquad \text{(8c)}$$

For the selected case

$x = C^2\theta_0 = \frac{1}{3}$, or $\theta_0 = 54.74°$, $1-y = S^2\phi_0 = (1+K^2)/2(3+K^2)$

For $K = 3$, $S^2\phi_0 = \frac{5}{12}$ or $\phi_0 = 40.20°$.

Hexad Analysis

Analysis of the selected hexad configuration is very similar to the pentad analysis except that a third nominally horizontal sensor, w, is added. This hexad thus also has fail/operational redundancy capability. Although hexad systems, with no three sensors coplanar, have been configured which can detect and isolate two successive failures for full fail/op-fail/op capability, Refs. 2 and 3 for example, this degree of redundancy is generally not required for Earth navigators and entails a penalty in accuracy and complexity compared with the selected one with three nominally horizontal axes and with θ_0 optimized.

Let six sensors be placed as shown in Fig. 7. Thus

$$u = p_1, \quad v = -p_1/2 + (3)^{1/2}r_1/2$$
$$w = -p_1/2 - (3)^{1/2}r_1/2$$
$$x = C\theta q + S\theta r_2$$
$$y = C\theta q + (3)^{1/2}S\theta p_2/2 - S\theta r_2/2$$
$$z = C\theta q - (3)^{1/2}S\theta p_2/2 - S\theta r_2/2$$

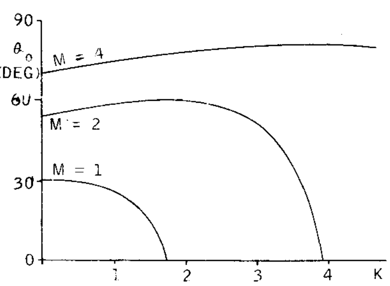

Fig. 4 θ_0 for hexad and tetrad.

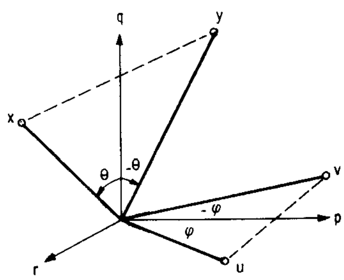

Fig. 6 Tetrad input axes.

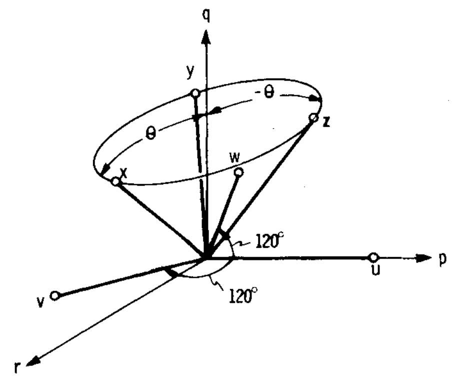

Fig. 7 Hexad input axes.

In this case we find

$$p = [(2u-v-w)+(3)^{1/2}S\theta(y-z)/(1+K^2C^2\theta)]/3[1+S^2\theta/(1+K^2C^2\theta)] \quad (7d)$$

$$r = [(3)^{1/2}(v-w)+S\theta(2x-y-z)/(1+K^2C^2\theta)]/3[1+S^2\theta/(1+K^2C^2\theta)] \quad (7e)$$

$$q = (x+y+z)/3C\theta \quad (7f)$$

Thus

$$\sigma_p{}^2 = \sigma_r{}^2 = 2\sigma_0{}^2/3[1+S^2\theta/(1+K^2C^2\theta)]$$

$$\sigma_q{}^2 = (1+K^2C^2\theta)\sigma_0{}^2/3C^2\theta$$

Minimizing $\sigma_N{}^2$ we find, as in the case of the tetrad

$$x = C^2\theta_0 = 1/[M(1+K^2)^{1/2}-(K^2-1)/2] \quad (8b)$$

θ_0 is plotted vs K for various M's on Fig. 4. Values of θ_0 for which $C^2\theta_0 = \frac{1}{3}$ form an attractive accurate system whose input axes can be viewed as being along the edges of a pyramid obtained by slicing off the corner of a cube. Such a system would be optimum for K and M which satisfy $M = (5+K^2)/2(1+K^2)^{1/2}$. One typical case for which $\cos^2\theta_0 = \frac{1}{3}$ is $K = 3$, $M = 7/(10)^{1/2} \simeq 2.2$.

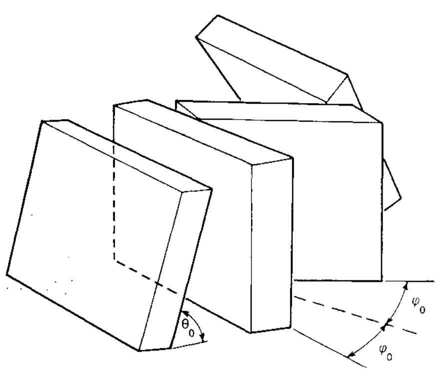

Fig. 8 Tetrad configuration.

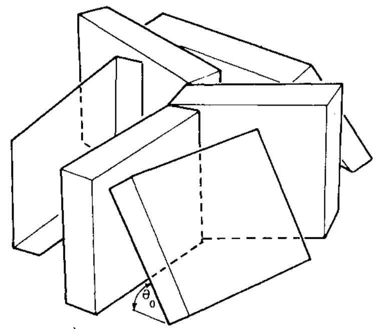

Fig. 9 Hexad configuration.

Accuracy Improvement

To evaluate the accuracy improvement possible with an optimized system, the system navigation variance for the above hexad case is

$$\sigma_{N_0}{}^2 = (1/2+1/2+1/2)\sigma_0{}^2/3 = 0.50\sigma_0{}^2$$

The corresponding variance for a hexad whose optimization neglects the effect of K and M (i.e., $\cos^2\theta = \frac{1}{3}$ for all six sensors) is

$$\sigma_N{}^2 = (1+K^2C^2\theta)[1/2+(10/49)(1/2)+1/2]\sigma_0{}^2/3$$

$$\sigma_N{}^2 = 4/3(1+5/49)\sigma_0{}^2 = 1.47\sigma_0{}^2$$

Thus, $\sigma_N{}^2$ is reduced by a factor of almost 3 for the new configuration. This dramatic improvement results from the fact that the three horizontal sensors have variances which are $\frac{1}{4}$ of that of the elevated sensors; also, because they are directed in the horizontal plane where information is more important at the expense of the azimuth axis where it is not needed as much as implied by $M \simeq 2.2$.

Figures 3, 8, and 9 are artist's conceptions of sensor arrays for the systems analyzed. The corresponding fault detection and isolation equations and the processing equations when a faulty sensor has been removed, are developed in the same way as described herein.

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